

Observables for identity-based tachyon vacuum solutions

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Abstract

We consider a modified KBc algebra in bosonic open string field theory expanded around identity-based scalar solutions. By use of the algebra, classical solutions on the background are constructed and observables for them, including energy densities and gauge invariant overlaps, are calculable. These results are applied to evaluate observables analytically for both of the identity-based trivial pure gauge solution and the identity-based tachyon vacuum solution.

§1. Introduction

An analytic tachyon vacuum solution was constructed on the basis of the identity string field, the BRST current, and the ghost field in bosonic cubic open string field theory.^{1),2),3)} The identity string field is a fundamental object in the open string field theory⁴⁾ and indeed it is a building block of the KBc algebra⁵⁾ by which wedge-based solutions⁶⁾ can be easily reconstructed. Then, the identity-based solutions were found by some left-right splitting algebra,¹⁾ which is similar to the KBc algebra in a sense, and a certain type of the identity-based solutions can be regarded as the tachyon vacuum solution. This is supported by evidence from study of the theory expanded around the solution: vanishing cohomology,^{2),3),7)} no open string excitations,⁸⁾ and the existence of the perturbative vacuum solution.^{9),10)} Hence, it appears highly probable that observables for the identity-based tachyon vacuum solution agree with those expected for the tachyon vacuum, although, due to characteristic subtleties of the identity string field, it has been difficult to perform direct evaluation of the observables.

Recently, significant progress has been made in the investigation of identity-based marginal solutions. We have obtained a gauge equivalence relation including the identity-based marginal solutions and some kind of wedge-based tachyon vacuum solutions and, using this relation, we can directly evaluate observables for the identity-based solutions.^{11),12)} The key ingredient is a combined technique for the identity-based solutions and the KBc algebra^{13),14)} and it has potentiality for investigating string field theory. In fact, it has been applied to construct a new solution,¹⁵⁾ which has the same algebraic structure as a wedge-based marginal solution¹⁶⁾ and is gauge equivalent to the identity-based marginal solution.

The main purpose of this paper is, based on these developments, to confirm directly that the identity-based scalar solution provides the correct observables as expected.

The identity-based scalar solution is given by¹⁾

$$\Psi_0 = Q_L(e^h - 1)I - C_L((\partial h)^2 e^h)I, \quad (1.1)$$

where $Q_L(f)$ and $C_L(f)$ are integrations of the BRST current $j_B(z)$ and the ghost $c(z)$, which are multiplied by a function $f(z)$ along a half unit circle. We find that the equation of motion holds for the function $h(z)$ such that $h(-1/z) = h(z)$ and $h(\pm i) = 0$. Moreover, the reality condition of (1.1) imposes the function $h(z)$ to satisfy $(h(z))^* = h(1/z^*)$.

Expanding the string field Ψ around the solution as $\Psi = \Psi_0 + \Phi$, we obtain an action for fluctuation:

$$S[\Psi; Q_B] = S[\Psi_0; Q_B] + S[\Phi; Q'], \quad (1.2)$$

where we denote the action as $S[\Psi; Q] = - \int (\frac{1}{2}\Psi * Q\Psi + \frac{1}{3}\Psi * \Psi * \Psi)$ and the kinetic operator Q' is given by

$$Q' = Q(e^h) - C((\partial h)^2 e^h). \quad (1.3)$$

The operators $Q(f)$ and $C(f)$ are defined as integrations along a whole unit circle.

We have a degree of freedom to choose a function $h(z)$ in the classical solution and it can be changed by gauge transformations. Since the function continuously connects to zero, most of the solutions are regarded as a trivial pure gauge solution. However, nontrivial solutions are generated at the boundary of some function spaces. In the well studied case, the function includes one parameter $a \geq -1/2$:

$$h_a(z) = \log \left(1 + \frac{a}{2}(z + z^{-1})^2 \right). \quad (1.4)$$

It is known that the solution for $a > -1/2$ is a trivial pure gauge, but it becomes a nontrivial solution for $a = -1/2$, for the reason mentioned above.

The transition from a trivial pure gauge to the tachyon vacuum solution has been observed in various aspects of the identity-based solution. In consequence, it is known that zeros of $e^{h(z)}$ move on the z plane with the deformation of $h(z)$ and then the transition occurs when the zeros reach the unit circle $|z| = 1$. For example, $e^{h(z)}$ for (1.4) is rewritten as

$$e^{h_a(z)} = \frac{1}{(1 - Z(a))^2} \{z^2 + Z(a)\} \{z^{-2} + Z(a)\} \left(Z(a) = \frac{1 + a - \sqrt{1 + 2a}}{a} \right), \quad (1.5)$$

and it has zeros at $\pm\sqrt{-Z(a)}$ and $\pm 1/\sqrt{-Z(a)}$. When the parameter a approaches $-1/2$ from positive infinity, $Z(a)$ runs from 1 to -1 and then it takes the value -1 for $a = -1/2$. As a result, we find that the zeros are on the unit circle only if $a = -1/2$ and then the solution becomes the tachyon vacuum solution.⁸⁾ For other functions, we find that the same transition occurs if the zeros move to the unit circle.³⁾

We now briefly outline our strategy. First, we find the KBc algebra in the shifted theory with the action $S[\Phi; Q']$, which we call the $K'Bc$ algebra. By means of the $K'Bc$ algebra, it is straightforward to construct classical solutions in the shifted theory and it is possible to calculate observables for these solutions. Here, the shifted theory includes one parameter a , as the above example, through Q' , and so the classical solutions depend on the parameter. This is similar to the case of the analysis for the identity-based marginal solutions,^{11),12)} in which the shifted theory and the solution include parameters related to marginal deformations. Therefore, according to the marginal case, we represent the identity-based solution as a gauge equivalence relation involving the identity-based and wedge-based

solutions. Finally, by use of this expression, we evaluate observables directly for the identity-based solution.

Later we will see that there is a difference between the $K'Bc$ algebra around the identity-based trivial solution and that around the identity-based nontrivial solution. If Ψ_0 is a trivial pure gauge solution, the $K'Bc$ algebra can be transformed to the original KBc algebra. With the help of the transformation, we can calculate observables for classical solutions in the shifted theory. Here, the existence of such a transformation depends crucially on the positions of zeros of the function $e^{h(z)}$. The zeros on the unit circle become obstacles to construction of the transformation and therefore it is impossible to transform from the $K'Bc$ algebra to the KBc one in the case that Ψ_0 is the tachyon vacuum solution. However, this implies that, on the identity-based tachyon vacuum, K' , B , and c have a different algebraic structure from the original KBc algebra. We will find that, on the identity-based tachyon vacuum, the operators K' and c commute with each other and then all the solutions made of K' , B and c can be written as modified BRST exact states. Accordingly, observables for them are calculable even if Ψ_0 is the tachyon vacuum solution.

This paper is organized as follows: First, we will consider classical solutions in the theory expanded around the identity-based solution in Sect. 2. We construct the $K'Bc$ algebra with respect to Q' and, by using the $K'Bc$ algebra, we will find classical solutions on the identity-based vacuum. To calculate observables for the classical solutions, we will construct a similarity transformation from the operator $(K'_1)_L$ to $(K_1)_L$. We will find that a conformal transformation is a significant part of the similarity transformation and so we will illustrate it by an example for $h_a(z)$ in (1.4). Then, we will calculate observables for the classical solutions around Ψ_0 . In Sect. 3, based on the results in the previous section, we analytically evaluate observables for the identity-based solutions. In Sect. 4, we will give concluding remarks. In Appendix A, we provide a detailed proof of the properties of a differential equation that plays an important role in the calculation of observables.

Note added: When we had a discussion with N. Ishibashi during the conference SFT2014 at SISSA, Trieste, it was found that we had reached the same conclusion for the gauge invariant observables for the identity-based tachyon vacuum solution.^{*)} The main difference is that he argued in detail for a regularization method to evaluate the observables¹⁷⁾ but we evaluated the observables for the identity-based trivial solution in addition to the tachyon vacuum case.

^{*)} Both Ishibashi's and our results were presented independently at the conference. The presentation files of these talks by N. Ishibashi and one of the authors (T.T.) are available on the conference website: <http://www.sissa.it/tpp/activity/conferences/SFT2014/>.

After almost completing the manuscript, we found a paper by S. Zeze,¹⁸⁾ which treats similar problems with different methods.

§2. Classical solutions around the identity-based solution

2.1. Modified KBc algebra

We can construct a modified KBc algebra associated with the deformed BRST operator (1·3):

$$K' = Q'B, \quad Q'K' = 0, \quad Q'c = cK'c, \quad (2.1)$$

$$B^2 = 0, \quad c^2 = 0, \quad Bc + cB = 1, \quad (2.2)$$

where B and c are the same string fields in the conventional KBc algebra,⁵⁾ and K' is given by^{**)}

$$K' = \frac{\pi}{2}(K'_1)_L I, \quad (K'_1)_L = \{Q', (B_1)_L\}. \quad (2.4)$$

The operator $(K'_1)_L$ is explicitly calculated as

$$(K'_1)_L = \int_{C_{\text{left}}} \frac{dz}{2\pi i} (1+z^2) \left\{ e^{h(z)} T(z) + (\partial h) e^{h(z)} j_{\text{gh}}(z) + \left(\frac{3}{2} \partial^2 h + \frac{1}{2} (\partial h)^2 \right) e^{h(z)} \right\}, \quad (2.5)$$

where $j_{\text{gh}}(z)$ is the ghost number current and $T(z)$ is the total energy momentum tensor.^{*)} We can easily find that if $h(z)$ becomes identically zero, the operator $(K'_1)_L$ is equal to the conventional $(K_1)_L$ in the KBc algebra.

For the general function $h(z)$, K' , B , c , and Q' have the same algebraic structure as that of the KBc algebra. However, if we choose a special function, the algebra is more simplified. To see this let us consider the relation $Q'c = cK'c$ in (2·1). This relation is derived from the following equations:

$$\{Q(e^h), c(z)\} = e^{h(z)} c \partial c(z), \quad (2.6)$$

$$[K'_1, c(z)] = -(\partial(1+z^2)) e^{h(z)} c(z) + (1+z^2) e^{h(z)} \partial c(z), \quad (2.7)$$

^{**)} We use the following convention:

$$B = \frac{\pi}{2}(B_1)_L I, \quad c = \frac{1}{\pi} c(1) I, \quad (B_1)_L = \int_{C_{\text{left}}} \frac{dz}{2\pi i} (1+z^2) b(z), \quad (2.3)$$

where I is the identity string field and the integration path C_{left} on the z -plane is a half unit circle: $|z| = 1$, $\text{Re } z \geq 0$.

^{*)} It can be calculated by using the relations in Ref. 7):

$$\{Q(f), b(z)\} = \frac{3}{2} \partial^2 f(z) + \partial f(z) j_{\text{gh}}(z) + f(z) T(z), \quad \{C(f), b(z)\} = f(z).$$

where K'_1 is the operator defined by the replacement of the integration path in (2.5) with a unit circle: $K'_1 = \{Q', B_1\} = \{Q', b_1 + b_{-1}\}$. As mentioned in Sect. 1, the function $e^{h(z)}$ has zeros on the unit circle in the case that the solution becomes the tachyon vacuum solution. Indeed, for (1.4), $e^{h_a(z)}$ has zeros at $z = \pm 1$ only in the case $a = -1/2$ and then, from (2.6), $Q(e^{h_a})$ and $c(1)$ anticommute with each other for $a = -1/2$.*) Similarly, K'_1 and $c(1)$ commute for $a = -1/2$ from (2.7). Consequently, we find a simplified algebra only in the case that $e^{h(z)}$ has zeros at $z = \pm 1$, namely in the theory around the tachyon vacuum solution:

$$K' = Q'B, \quad Q'K' = 0, \quad Q'c = 0, \quad (2.8)$$

$$B^2 = 0, \quad c^2 = 0, \quad Bc + cB = 1. \quad (2.9)$$

Actually, there are other possibilities²⁾ where the function $e^{h(z)}$ for the tachyon vacuum solution has zeros on the unit circle but not at $z = \pm 1$. We will discuss these cases at the end of the section.

2.2. Classical solutions

The equation of motion in the theory around the solution (1.1) is given by

$$Q'\Phi + \Phi^2 = 0, \quad (2.10)$$

where Q' is the modified BRST operator (1.3). We can find various classical solutions in the shifted background by substituting K' for K in the solutions given by the KBc algebra in the original theory. In the conventional theory with Q_B , a classical solution using the KBc algebra is written as

$$\Psi_0(K, B, c) = \sum_{ij} \mathcal{A}_i(K) c \mathcal{B}_j(K) + \sum_{ijk} \mathcal{C}_i(K) c \mathcal{D}_j(K) c \mathcal{E}_k(K) B, \quad (2.11)$$

which is general configuration with ghost number one in terms of the KBc algebra. Here, $\mathcal{A}_i(K)$, $\mathcal{B}_i(K)$, $\mathcal{C}_i(K)$, $\mathcal{D}_i(K)$, and $\mathcal{E}_i(K)$ are appropriate functions of the string field K . Once a particular solution (2.11) is given, a classical solution for (2.10) is constructed as

$$\Phi_0(K', B, c) = \sum_{ij} \mathcal{A}_i(K') c \mathcal{B}_j(K') + \sum_{ijk} \mathcal{C}_i(K') c \mathcal{D}_j(K') c \mathcal{E}_k(K') B. \quad (2.12)$$

If $\Psi_0(K, B, c)$ is a solution in the conventional theory, $\Phi_0(K', B, c)$ is a solution in the shifted background, regardless of whether or not the $K'Bc$ algebra is simplified as (2.8). However, we emphasize that in the case that the algebra is simplified, the solution has a simpler expression:

$$\Phi_0(K', c) = \mathcal{F}(K')c, \quad (2.13)$$

*) We note that $e^{-i\sigma}c(e^{i\sigma})I = -e^{-i(\pi-\sigma)}c(e^{i(\pi-\sigma)})I$ and therefore $c(1)I = c(-1)I$.

where $\mathcal{F}(K') = \sum_{ij} \mathcal{A}_i(K') \mathcal{B}_j(K')$ and the second term in (2.12) vanishes due to $K'c = cK'$ and $c^2 = 0$.

2.3. Transformations from $(K'_1)_L$ to $(K_1)_L$

In this subsection, we will consider a similarity transformation from $(K'_1)_L$ to the conventional $(K_1)_L$.

First, we introduce the operator^{1), 8)*)}

$$\tilde{q}(h) = \oint \frac{dz}{2\pi i} h(z) \left(j_{\text{gh}}(z) - \frac{3}{2} z^{-1} \right), \quad (2.14)$$

where $j_{\text{gh}}(z)$ is the ghost number current, $j_{\text{gh}} = cb$. Using this operator, the modified BRST operator (1.3) is transformed to the original BRST operator:^{1)**)}

$$e^{-\tilde{q}(h)} Q' e^{\tilde{q}(h)} = Q_B. \quad (2.15)$$

Accordingly, we can remove the ghost number current from $(K'_1)_L$ in (2.5) by a similarity transformation:

$$\begin{aligned} e^{-\tilde{q}(h)} (K'_1)_L e^{\tilde{q}(h)} &= e^{-\tilde{q}(h)} \{Q', (B_1)_L\} e^{\tilde{q}(h)} = \{Q_B, e^{-\tilde{q}(h)} (B_1)_L e^{\tilde{q}(h)}\} \\ &= \int_{C_{\text{left}}} \frac{dz}{2\pi i} (1+z^2) e^{h(z)} T(z), \end{aligned} \quad (2.16)$$

where we have used $e^{-\tilde{q}(h)} b(z) e^{\tilde{q}(h)} = e^{h(z)} b(z)$.⁸⁾

Next, we look for a conformal transformation $z' = f(z)$ that maps (2.16) to $(K_1)_L$. Since $T(z)$ is a primary field with the dimension 2, the operator (2.16) is transformed as

$$f \left[\int_{C_{\text{left}}} \frac{dz}{2\pi i} (1+z^2) e^{h(z)} T(z) \right] = \int_{C'_{\text{left}}} \frac{df}{2\pi i} (1+z^2) e^{h(z)} \frac{df}{dz} T(f(z)), \quad (2.17)$$

where C'_{left} is an integration path in the mapped plane such as $f : C_{\text{left}} \rightarrow C'_{\text{left}}$. In order that (2.17) coincides with $(K_1)_L$, the function $f(z)$ must satisfy a differential equation:

$$(1+z^2) e^{h(z)} \frac{df}{dz} = 1 + f^2, \quad (2.18)$$

and C'_{left} must remain the same path along the left half of a string.

To find the conformal map, it is necessary to solve the differential equation (2.18) in an annulus including the unit circle $|z| = 1$. The important point is that we can solve it if $e^{h(z)}$

*) This operator was written as $K(h)$ in Ref. 8). The ghost number current $j_{\text{gh}}(z)$ is defined by using $SL(2, \mathbb{R})$ normal ordering. If $h(z)$ satisfies $h(-1/z) = h(z)$, owing to the second term $(-3/2z^{-1})$, the operator is transformed as $\tilde{q}(h) \rightarrow -\tilde{q}(h)$ under the BPZ conjugation. Moreover, $\tilde{q}(h)$ is a derivation with respect to the star product among string fields.

**) The operator $e^{\pm \tilde{q}(h)}$ becomes singular for the tachyon vacuum solution.¹⁾

has no zeros on the unit circle, as seen in Appendix A. Moreover, we can prove that under the initial condition $f(1) = 1$, the solution $f(z)$ has the following properties:

$$1. \quad |z| = 1 \Rightarrow |f(z)| = 1, \quad (2.19)$$

$$2. \quad f : C_{\text{left}} \rightarrow C_{\text{left}}, \quad (2.20)$$

$$3. \quad f\left(-\frac{1}{z}\right) = -\frac{1}{f(z)}. \quad (2.21)$$

We illustrate these by the solution given for $h_a(z)$ (1.4) in the next subsection and we give a detailed proof in Appendix A.

From (2.19) and (2.20), we find that the conformal map by the solution f leaves the integration path in (2.17) unchanged, namely $C'_{\text{left}} = C_{\text{left}}$. Therefore we can transform the operator (2.16) to $(K_1)_L$ by the conformal transformation f . Moreover, (2.21) indicates that the conformal map $f(z)$ is generated by the operators $K_n = L_n - (-1)^n L_{-n}$.^{*)}

Consequently, around the trivial pure gauge solution, we can construct the similarity transformation

$$U_f e^{-\tilde{q}(h)} (K'_1)_L e^{\tilde{q}(h)} U_f^{-1} = (K_1)_L, \quad (2.22)$$

where U_f is the operator for the conformal transformation f and it is given in the form

$$U_f = \exp \left(\sum_n v_n K_n \right), \quad (2.23)$$

with certain parameters v_n .

For the identity-based tachyon vacuum solution, a solution to the differential equation (2.18) for $f(z)$ has singularity due to zeros of $e^{h(z)}$ on the unit circle (Appendix A). In this sense, we emphasize that a regular operator U_f does not exist for the tachyon vacuum.

2.4. An example for the transformation

We illustrate the existence of the transformation U_f by solving (2.18) for (1.4). For (1.4), $e^{h(z)}$ is written as (1.5), and, under the initial condition $f(1) = 1$, setting $z = e^{i\sigma}$, we can solve the differential equation (2.18) as follows:

$$f(e^{i\sigma}) = e^{i\phi(\sigma)}, \quad \phi(\sigma) = \sigma + 2 \arctan \frac{g(\sigma) \cos \sigma}{1 + g(\sigma) \sin \sigma}, \quad (2.24)$$

where, for $-1/2 < a \leq 0$ ($-1 < Z(a) \leq 0$), $g(\sigma)$ is given as

$$g(\sigma) = \tanh \left\{ \frac{\sqrt{-Z(a)}}{1 + Z(a)} \arctan \left(\frac{2\sqrt{-Z(a)}}{1 + Z(a)} \sin \sigma \right) \right\}, \quad (2.25)$$

^{*)} K_n generates a transformation $f(\sigma)$ such that $f(\pi - \sigma) = \pi - f(\sigma)$.⁴⁾ By setting $z = e^{i\sigma}$, this corresponds to (2.21).

and, for $0 < a$ ($0 < Z(a) < 1$), it is

$$g(\sigma) = -\tanh \left\{ \frac{\sqrt{Z(a)}}{1+Z(a)} \operatorname{arctanh} \left(\frac{2\sqrt{Z(a)}}{1+Z(a)} \sin \sigma \right) \right\}. \quad (2.26)$$

Since $\phi(\sigma)$ is a real-valued function for $\sigma \in \mathbb{R}$, the solution (2.24) satisfies (2.19). By differentiating $\phi(\sigma)$, it can be seen that $\phi(\sigma)$ is a monotonically increasing function for $-\pi/2 < \sigma < \pi/2$. We also see that $\phi(\pm\pi/2) = \pm\pi/2$. Therefore, C'_{left} is the same as the left half of a string, then (2.20) is satisfied. Moreover, since $g(\pi - \sigma) = g(\sigma)$, we find that $\phi(\pi - \sigma) = \pi - \phi(\sigma)$ and then the function (2.24) satisfies (2.21). Thus, the solution (2.24) satisfies (2.19), (2.20) and (2.21) in the case of $a > -1/2$, and then the transformation U_f exists.

Here, we should emphasize that the transformation (2.23) exists only in the case $a > -1/2$ and it does not at $a = -1/2$, because the circle-to-circle correspondence for the integration path is broken down for $a = -1/2$. In fact, taking the limit $a \rightarrow -1/2$, the phase $\phi(\sigma)$ in (2.24) approaches a step function:

$$\lim_{a \rightarrow -1/2} \phi(\sigma) = \begin{cases} \frac{\pi}{2} & (0 < \sigma < \pi) \\ -\frac{\pi}{2} & (-\pi < \sigma < 0) \end{cases}. \quad (2.27)$$

Therefore, we cannot transform $(K'_1)_L$ to $(K_1)_L$ by a regular conformal map at $a = -1/2$.

2.5. Observables around the trivial pure gauge solution

In this subsection, we will show that observables for the solution (2.12) around the identity-based trivial pure gauge solution are equivalent to those for the original solution (2.11).

First, we find that $c(1)$ and $(B_1)_L$ are invariant under the similarity transformation (2.22), namely, $U_f e^{-\tilde{q}(h)} c(1) e^{\tilde{q}(h)} U_f^{-1} = c(1)$ and $U_f e^{-\tilde{q}(h)} (B_1)_L e^{\tilde{q}(h)} U_f^{-1} = (B_1)_L$. Using $e^{-\tilde{q}(h)} c(z) e^{\tilde{q}(h)} = e^{-h(z)} c(z)$,⁸⁾ we have

$$U_f e^{-\tilde{q}(h)} c(z) e^{\tilde{q}(h)} U_f^{-1} = e^{-h(z)} \left(\frac{df(z)}{dz} \right)^{-1} c(f(z)). \quad (2.28)$$

From the differential equation (2.18), it follows that

$$= \frac{1+z^2}{1+f(z)^2} c(f(z)), \quad (2.29)$$

and then $c(1)$ is invariant under the transformation because $f(1) = 1$ is imposed as the initial condition. With regard to $(B_1)_L$, the invariance can be easily seen by using $e^{-\tilde{q}(h)} b(z) e^{\tilde{q}(h)} = e^{h(z)} b(z)$ ⁸⁾ and the fact that $b(z)$ is a primary field with the dimension 2.

Now that the similarity transformation of $(K'_1)_L$, $(B_1)_L$, and $c(1)$ is established, we can transform the solution (2.12) to the original solution (2.11). An important point is that the generators $\tilde{q}(h)$ and K_n are derivations with respect to the star product and in particular $\tilde{q}(h)I = 0$ and $K_n I = 0$. Then, we obtain the transformation from string fields (K, B, c) to (K', B, c) :

$$K' = e^{\tilde{q}(h)} U_f^{-1} K, \quad B = e^{\tilde{q}(h)} U_f^{-1} B, \quad c = e^{\tilde{q}(h)} U_f^{-1} c. \quad (2.30)$$

Noting that U_f^{-1} and $e^{\tilde{q}(h)}$ are given as an exponential of derivations, we find that the solution (2.12) is given as a transformation from (2.11):

$$\Phi_0(K', B, c) = e^{\tilde{q}(h)} U_f^{-1} \Psi_0(K, B, c). \quad (2.31)$$

Let us consider the vacuum energy for Φ_0 around the trivial pure gauge solution. Using the transformation (2.31), the action for Φ_0 is given by

$$S[\Phi_0(K', B, c); Q'] = S[\Psi_0(K, B, c); U_f Q_B U_f^{-1}], \quad (2.32)$$

where we have used (2.15) and the BPZ conjugation: $e^{\tilde{q}(h)} U_f^{-1} \rightarrow U_f e^{-\tilde{q}(h)}$. Since U_f is generated by K_n and the operators Q_B and L_n commute with each other, $U_f Q_B U_f^{-1}$ is equal to Q_B . As a result, the vacuum energy for $\Phi_0(K', B, c)$ is equivalent to that for $\Psi_0(K, B, c)$ in the conventional theory.

Next, let us consider gauge invariant overlaps for $\Phi_0(K', B, c)$. The gauge invariant overlap for the open string field Ψ is defined as¹⁹⁾

$$O_V(\Psi) = \langle I | V(i) | \Psi \rangle, \quad (2.33)$$

where $V(i)$ is a closed string vertex operator, such as $c(i)c(-i)V_m(i, -i)$, where $V_m(z, \bar{z})$ is a matter primary with the conformal dimension $(1, 1)$. Noting that $h(\pm i) = 0$, in spite of the closed string vertex on I , $\tilde{q}(h)$ satisfies

$$\langle I | V(i) \tilde{q}(h) = 0. \quad (2.34)$$

In addition, the operators K_n generate a global symmetry of the open string field theory even if the gauge invariant overlaps are introduced as sources. In fact, since $V(i)$ has the dimension 0 and $f(\pm i) = \pm i$, we find that $U_f V(i) U_f^{-1} = V(f(i)) = V(i)$ and then

$$\langle I | V(i) U_f^{-1} = \langle I | V(i). \quad (2.35)$$

Consequently, we can see that the gauge invariant overlaps for the solution (2.12) are equivalent to that for the conventional solution (2.11):^{20), 21)}

$$O_V(\Phi_0(K', B, c)) = O_V(\Psi_0(K, B, c)). \quad (2.36)$$

2.6. Observables around the tachyon vacuum solution

Now let us consider observables for the classical solution (2.13) around the identity-based tachyon vacuum solution. In this vacuum, the modified $K'Bc$ algebra and the classical solution are simplified as mentioned before, and Q' has vanishing cohomology.^{2),7)} From $Q'c = 0$ in (2.8), c turns out to be an exact state with respect to the modified BRST operator Q' . Since $Q'K' = 0$, the solution (2.13) can be written as a modified BRST exact state:

$$\Phi_0(K', c) = Q'\chi. \quad (2.37)$$

Therefore, we conclude that both the vacuum energy and the gauge invariant overlaps are zero for the classical solution (2.13).

Here, we should note that the derivation of (2.37) requires careful consideration. In Ref. 7), the homotopy operator is given for the BRST operator Q' in the identity-based tachyon vacuum. In the case of the tachyon vacuum solution using the function (1.4) with $a = -1/2$, a corresponding homotopy operator is $\hat{A} = (b(1) + b(-1))/2$. If it is used for the above exact form, such as $\chi = \mathcal{F}(K')\hat{A}c$, the divergence arises from a collision between $b(1)$ and $c(1)$ and so we need regularization of (2.13). However, it should be noted that the homotopy operator \hat{A} , such as $\{\hat{A}, Q'\} = 1$, is not unique because we can add to it a commutator $[Q', \mathcal{O}]$, where \mathcal{O} is an arbitrary operator with ghost number -2 . Then, we have the possibility of providing a regularization procedure by adding such terms to the homotopy operator. In Ref. 17), several regularization methods are rigorously discussed.

In addition, we should notice that the divergence does not appear in the procedure used to analyze the cohomology of Q' in Ref. 2). The string field c can be expanded in the Fock space for each L_0 -level and the lowest-level state in c is $c_1|0\rangle$:

$$c = \frac{1}{2\pi}c_1|0\rangle + \cdots. \quad (2.38)$$

For the identity-based tachyon vacuum using the function (1.4) with $a = -1/2$, Q' has an oscillator expression:

$$Q' = R_2 + R_0 + R_{-2}, \quad (2.39)$$

where R_n stands for terms with the mode number n with respect to L_0 .^{*)}

$$R_{\pm 2} = -\frac{1}{4}Q_{\pm 2} + c_{\pm 2}, \quad R_0 = \frac{1}{2}Q_B + 2c_0. \quad (2.40)$$

^{*)} Here, we have expanded the conventional primary BRST current j_B as $j_B(z) = \sum_{n=-\infty}^{\infty} Q_n z^{-n-1}$ and therefore $Q_0 = Q_B$ in particular. The nilpotency of Q' leads to the anticommutation relations,

$$\{R_{\pm 2}, R_{\pm 2}\} = 0, \quad \{R_{\pm 2}, R_0\} = 0, \quad 2\{R_2, R_{-2}\} + \{R_0, R_0\} = 0.$$

Then, $Q'c$ is written by the Fock space state starting from a lowest-level state:

$$Q'c = \frac{1}{2\pi} R_2 c_1 |0\rangle + \cdots. \quad (2.41)$$

Therefore, we can solve the equation $Q'c = 0$ level by level and then c can be given by a modified BRST exact state with no divergence, because, as in Ref. 2), the cohomology of Q' is expressed by the well-defined Fock space expression. In particular, there is no cohomology within the ghost number one sector. Thus, by solving the cohomology level by level, we can write c as a Q' exact state with a well-defined Fock space expression. As a result, the expression (2.37) can be well-defined with no divergence.

2.7. Comments on a simplified algebra

Here, we comment on the case that $e^{h(z)}$ for the identity-based solution has zeros on the unit circle but not at $z = \pm 1$. In Ref. 2), the identity-based solutions (1.1) with the function $h_a^l(z)$ ($l = 1, 2, 3, \dots; a \geq -1/2$):

$$h_a^l(z) = \log \left(1 - \frac{a}{2} (-1)^l (z^l - (-z^{-1})^l)^2 \right) \quad (2.42)$$

were considered as a generalization of the function $h_a(z)$ (1.4), which is the case of $l = 1$ in the above. The solution corresponding to $h_a^l(z)$ is pure gauge for $a > -1/2$ and we can apply the prescriptions in the previous subsections. In the case that $a = -1/2$, the corresponding solution is believed to represent the tachyon vacuum, where the BRST operator Q' around the solution has no cohomology,^{2),7)} and we have

$$e^{h_{-1/2}^l(z)} = \frac{(-1)^l}{4} (z^l + (-z^{-1})^l)^2. \quad (2.43)$$

It has zeros at $z = \pm 1$ when l is a positive odd integer and we can use simplified algebra (2.8) in the same way as the case of the function (1.4).

In the case that $l = 2m$ ($m = 1, 2, \dots$), i.e. a positive even integer, the function (2.43) has zeros on the unit circle: $z_k = e^{i\theta_k}$, where $\theta_k = \frac{2k-1}{4m}\pi$ ($k = 1, 2, \dots, 4m$), and they are not ± 1 . In this case, the simplified $K'Bc$ algebra (2.8) does not hold because $e^{h_{-1/2}^{2m}(1)} \neq 0$. However, we can obtain a simplified algebra by using

$$c' = \frac{1}{\pi \cos \theta_1} e^{-i\theta_1} c(e^{i\theta_1}) I \quad (2.44)$$

with $\theta_1 = \frac{\pi}{4m}$ instead of $c = \frac{1}{\pi} c(1) I$ as follows. Firstly, we note that

$$e^{\alpha K_1} c = \frac{2}{\pi} U_1^\dagger U_1 \tilde{c}(\alpha) |0\rangle, \quad (2.45)$$

where $\tilde{c}(\tilde{z}) = \tan \circ c(\tilde{z})$ and we have used the notation in Ref. 6). Using a relation

$$(U_1^\dagger U_1)^{-1} c(e^{i\theta})(U_1^\dagger U_1) = \left(\cos(it + \frac{\pi}{4}) \right)^{-2} \tilde{c}(it), \quad e^{i\theta} = \tan(it + \frac{\pi}{4}), \quad (2.46)$$

we have

$$e^{itK_1} c = \frac{2}{\pi} \cos^2(it + \frac{\pi}{4}) c(e^{i\theta}) I = \frac{1}{\pi \cos \theta} e^{-i\theta} c(e^{i\theta}) I. \quad (2.47)$$

Therefore, with t_1 such as $e^{it_1} = \tan(it_1 + \frac{\pi}{4})$, or $t_1 = \operatorname{arctanh}(\tan \frac{\theta_1}{2})$, c' defined in (2.44) can be expressed as $c' = e^{it_1 K_1} c$. Because $K_1 = L_1 + L_{-1}$ is a derivation with respect to the star product, and noting $[K_1, (B_1)_L] = 0$ and $[K_1, Q_B] = 0$, we have

$$Q_B c' = e^{it_1 K_1} Q_B c = c' K c', \quad B c' + c' B = e^{it_1 K_1} (B c + c B) = 1, \quad (c')^2 = e^{it_1 K_1} c^2 = 0, \quad (2.48)$$

and they form a kind of $K B c'$ algebra. Furthermore, noting (2.6) and (2.7), we obtain a simplified algebra

$$K' = Q' B, \quad Q' K' = 0, \quad Q' c' = 0, \quad (2.49)$$

$$B^2 = 0, \quad (c')^2 = 0, \quad B c' + c' B = 1, \quad (2.50)$$

for a modified BRST operator Q' corresponding to the function $h_{-1/2}^{2m}(z)$. Using the above, we can apply the prescription in Sect. 2.6 in a similar way.

§3. Observables for identity-based solutions

We consider direct calculation of observables for the identity-based tachyon vacuum solutions, by use of the method for the identity-based marginal solution in Ref. 11).

We consider one parameter family of the identity-based solution, $\Psi_0(a)$. The parameter a deforms the function $h(z)$ in the solution, and as the simplest case (1.4) it takes values $a \geq -1/2$, the solution becomes the tachyon vacuum at $a = -1/2$,^{*)} otherwise it is a trivial solution. In particular, we assume that $\Psi_0(a = 0) = 0$.

Suppose that $\Psi_0(K, B, c)$ in (2.11) is a tachyon vacuum solution in the conventional theory. Then, $\Phi_0(K', B, c)$ in (2.12) is a tachyon vacuum solution in the theory with Q' for $a > -1/2$, but, in the case of $a = -1/2$, $\Phi_0(K', c)$ in (2.13) is a trivial pure gauge solution.

Here, we take $\Psi_a = \Psi_0(a) + \Phi_0$ for $a \geq -1/2$. We can easily find that Ψ_a is a classical solution in the conventional theory, namely it satisfies $Q_B \Psi_a + \Psi_a^2 = 0$. Expanding the string field around Ψ_a in the action, we have the kinetic operator Q_{Ψ_a} : $Q_{\Psi_a} A = Q_B A + \Psi_a A - (-1)^{|A|} A \Psi_a$ for all string field A . Q_{Ψ_a} can be written as

$$Q_{\Psi_a} A = Q' A + \Phi_0 A - (-1)^{|A|} A \Phi_0 = Q'_{\Phi_0} A, \quad (3.1)$$

^{*)} Namely, $e^{h(z)}$ has zeros on the unit circle at $a = -1/2$.

where Q' is the modified BRST operator in (1.3) and Q'_{Φ_0} represents the kinetic operator around the solution Φ_0 in the theory at the identity-based vacuum $\Psi_0(a)$. The important point is that we can construct a homotopy operator for $Q'_{\Phi_0}(=Q_{\Psi_a})$ for $a \geq -1/2$ by use of $K'Bc$ algebra.^{*)}

Differentiating the equation of motion, $Q_B \Psi_a + \Psi_a^2 = 0$, with respect to a , we find

$$Q_{\Psi_a} \frac{d}{da} \Psi_a = 0. \quad (3.2)$$

Since Q_{Ψ_a} has vanishing cohomology, we have

$$\frac{d}{da} \Psi_a = Q_{\Psi_a} \Lambda_a, \quad (3.3)$$

for some state Λ_a . Integrating (3.3) from $a = 0$, we get

$$\Psi_0(a) + \Phi_0 = \Psi_0(K, B, c) + \int_0^a Q_{\Psi_a} \Lambda_a da, \quad (3.4)$$

where we have used the fact that in the case of $a = 0$, $\Psi_0(a = 0) = 0$ and the $K'Bc$ solution is the same as the conventional tachyon vacuum solution: $\Phi_0(K', B, c) = \Psi_0(K, B, c)$.

From (3.4), we can calculate the gauge invariant overlap for the identity-based solution:

$$O_V(\Psi_0(a)) = O_V(\Psi_0(K, B, c)) - O_V(\Phi_0), \quad (3.5)$$

where we have used the fact that the gauge invariant overlap is BRST invariant with respect to Q_{Ψ_a} : $O_V(Q_{\Psi_a}(\dots)) = 0$. Noting that the formula (3.5) holds for $a \geq -1/2$, by using the result of the gauge invariant overlap for Φ_0 in the previous section, the above is evaluated as

$$O_V(\Psi_0(a)) = \begin{cases} 0 & (a > -1/2) \\ \frac{1}{\pi} \left\langle V(i\infty) c(\frac{\pi}{2}) \right\rangle_{C_\pi} & (a = -1/2) \end{cases}, \quad (3.6)$$

where we have used the notation in Ref. 11). Thus, as expected for the identity-based solution, the gauge invariant overlap for $a > -1/2$ is equal to that of trivial pure gauge solutions, and, in the case that $a = -1/2$, the gauge invariant overlap agrees with the result for the tachyon vacuum solution.

As emphasized in Ref. 12), the formula (3.4) is nothing but a gauge equivalence relation between $\Psi_0(a) + \Phi_0$ and $\Psi_0(K, B, c)$. In fact, given the relation (3.4), $\Psi_0(a) + \Phi_0$ can be written as

$$\Psi_0(a) + \Phi_0 = g^{-1} Q_B g + g^{-1} \Psi_0(K, B, c) g, \quad (3.7)$$

^{*)} In the case that $\Psi_0(K, B, c)$ is the Erler-Schnabl solution,²²⁾ e.g., we have $\Phi_0(K', B, c) = \frac{1}{\sqrt{1+K'}}(c + cK'Bc)\frac{1}{\sqrt{1+K'}}$ and the corresponding homotopy operator is given by a homotopy state: $\frac{1}{\sqrt{1+K'}}B\frac{1}{\sqrt{1+K'}}$ in the same way as Ref. 13).

where g is given by the path-ordered exponential form,

$$g = \text{P exp} \left(\int_0^a \Lambda_a da \right). \quad (3.8)$$

From this gauge equivalence relation, we have

$$S[\Psi_0(a); Q_B] + S[\Phi_0; Q'] = S[\Psi_0(K, B, c); Q_B]. \quad (3.9)$$

From the result for $S[\Phi_0; Q']$ in the previous section and for the conventional tachyon vacuum, namely $S[\Psi_0(K, B, c); Q_B] = 1/(2\pi^2)$, we finally find that

$$-S[\Psi_0(a); Q_B] = \begin{cases} 0 & (a > -1/2) \\ -\frac{1}{2\pi^2} & (a = -1/2) \end{cases}. \quad (3.10)$$

Thus, we have evaluated the vacuum energy density for the identity-based solution and these results are consistent with our expectation for the solution.

§4. Concluding remarks

We have constructed classical solutions Φ_0 in the theory expanded around the identity-based scalar solution Ψ_0 (1.1). We have taken advantage of the $K'Bc$ algebra to calculate observables for the solution. In the case that Ψ_0 is trivial pure gauge, the observables for Φ_0 are equivalent to those for the corresponding solution $\Psi_0(K, B, c)$ in the original background. In the case that Ψ_0 is the tachyon vacuum, they become equal to those for trivial solutions, since the $K'Bc$ algebra is simplified and all the solutions made from K' , B , and c are given as Q' -exact states. Finally, we have provided the gauge equivalence relation between $\Psi_0(a) + \Phi_0$ and $\Psi_0(K, B, c)$, which is regarded as a new expression for the identity-based solution. Thanks to this expression, we have analytically calculated observables for the identity-based scalar solution whether it corresponds to trivial pure gauge or tachyon vacuum.

Around the identity-based tachyon vacuum solution, the zeros of $e^{h(z)}$ on the unit circle play a crucial role in evaluating observables for Φ_0 . As seen in Sect. 2.7, there is no need for these zeros to be at $z = \pm 1$, which correspond to open string boundaries. We note that we can find similar results in the study of homotopy operators for the BRST operator around the identity-based scalar solutions,^{7),23)} in which homotopy operators exist only if the zeros are on the unit circle. Here, we should comment on another identity-based solution discussed in Ref. 3), in which $e^{h(z)}$ has higher-order zeros than the function in this paper. However,

similar to the discussion of homotopy operators in Ref. 7), we can obtain the simplified $K'Bc$ algebra with higher-order zeros and an important point is the position of the zeros rather than the order.

For the simplest function $h_a(z)$ (1.4), the solution $\Phi_0(K', B, c)$ in the theory around $\Psi_0(a)$ depends on the parameter a . We find that for $a > -1/2$, Φ_0 can correspond to the tachyon vacuum but for $a = -1/2$, it becomes a trivial pure gauge configuration as stated in Sect. 2.6. This result is in accordance with the numerical analysis in Ref. 24), where it is observed that in the theory around $\Psi_0(a > -\frac{1}{2})$, we can construct a numerical solution whose energy density corresponds to the negative of the D-brane tension, while the solution continuously connects to trivially zero as a approaches $-1/2$. We find that the transition becomes sharp if the truncation level increases. Accordingly, Φ_0 can be identified as the numerical solution in Ref. 24) although they belong to different gauge sectors. Here, we should note that, for $a = -1/2$, we expect that there exists a solution whose energy density is the *positive* of the D-brane tension. Such a solution has been constructed numerically in the Siegel gauge in Refs. 9), 10). It should represent the perturbative vacuum where a D-brane exists.*)

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Appendix A

—— Solutions for the differential equation (2.18) ——

Let us consider solutions of (2.18). It is sufficient to solve the equation in an annulus including the unit circle $|z| = 1$, because now we look for a regular function $f(z)$ on the circle.

The differential equation (2.18) is reducible to the homogeneous equation,

$$\frac{dg(z)}{dz} - 2iX(z)g(z) = 0, \quad X(z) = -\frac{1}{(1+z^2)e^{h(z)}}, \quad (\text{A.1})$$

*) N. Ishibashi pointed out the possibility of constructing the perturbative vacuum solution in a private discussion.¹⁷⁾

by the variable transformation,

$$f(z) = i \frac{2g(z) + i}{2g(z) - i}. \quad (\text{A}\cdot 2)$$

Solving Eq. (A.1) under the initial condition $f(1) = 1$, the function $f(z)$ is given by

$$f(z) = -i \frac{1 + ie^{v(z)}}{1 - ie^{v(z)}}, \quad v(z) = 2i \int_1^z X(z') dz'. \quad (\text{A}\cdot 3)$$

Since the function $X(z)$ has singularity at $z = \pm i$, $v(z)$ becomes divergent and so the expression (A.3) is undefined at the midpoints. Here, let us analyze the behavior of $f(z)$ near the midpoints in terms of series solutions. Suppose that $h(z)$ is holomorphic at $z = i$. Since $X(z)$ has a single pole at $z = i$, $X(z)$ is expanded into a Laurent series:

$$X(z) = \frac{1}{z - i} \sum_{n=0}^{\infty} x_n (z - i)^n, \quad (\text{A}\cdot 4)$$

where the first few coefficients are given by

$$x_0 = \frac{ie^{-h(i)}}{2}, \quad x_1 = -\frac{e^{-h(i)}}{4}(1 + 2i\partial h(i)), \quad \dots \quad (\text{A}\cdot 5)$$

Using this expansion, we can construct a series solution for the differential equation (A.1):

$$g(z) = (z - i)^\lambda \sum_{n=0}^{\infty} A_n (z - i)^n, \quad (\text{A}\cdot 6)$$

where we find that $A_0 \neq 0$, $\lambda = -e^{-h(i)}$, and other A_n are given by a recurrence formula:

$$A_n = \frac{2i}{n} (x_1 A_{n-1} + x_2 A_{n-2} + \dots + x_n A_0). \quad (\text{A}\cdot 7)$$

It can be easily seen that this series solution is convergent in a neighborhood of $z = i$. Since $\lambda = -1$ due to $h(\pm i) = 0$ for the identity-based solution, $f(z)$ is a holomorphic function near $z = i$:

$$f(z) = i \frac{2(z - i)g(z) + i(z - i)}{2(z - i)g(z) - i(z - i)}. \quad (\text{A}\cdot 8)$$

Taking the limit $z \rightarrow i$, we find that $f(i) = iA_0/A_0 = i$. In addition, we have $f'(i) = -1/A_0 \neq 0$. Similarly, we find that $f(z)$ is holomorphic at $z = -i$, and that $f(-i) = -i$ and $f'(-i) \neq 0$.

We have found that the poles of $X(z)$ at $z = \pm i$ are harmless to solve (2.18). However, if $X(z)$ has poles on the unit circle due to zeros of $e^{h(z)}$, it is difficult to find regular solutions on the unit circle. Suppose that $X(z)$ is expanded around the zero $z_0 (\neq \pm i, |z_0| = 1)$ as

$$X(z) = \frac{x'_0}{z - z_0} + x'_1 + \dots, \quad (\text{A}\cdot 9)$$

$g(z)$ behaves around $z = z_0$ as

$$g(z) \sim (z - z_0)^{2ix'_0} \times (\dots), \quad (\text{A}\cdot 10)$$

where the dots denote a regular function. Here it is noted that $X(z)$ has poles on the unit circle, but the residue x'_0 is essentially unrestricted as opposed to the residue x_0 at $z = \pm i$. Therefore, $g(z)$ is not a regular function in general and so it is impossible to find a regular conformal transformation $f(z)$ if $e^{h(z)}$ has zeros on the unit circle. Actually, we have seen an example for a singular map in Sect. 2.4.

Now, let us consider (2.19) and (2.20) for the solution (A.3). For $z = e^{i\sigma}$, ($|z| = 1$), $v(z)$ is written by

$$v(e^{i\sigma}) = \int_0^\sigma \frac{1}{e^{h(e^{i\sigma})} \cos \sigma} d\sigma. \quad (\text{A}\cdot 11)$$

As mentioned in the introduction, the reality condition of Ψ_0 implies $(h(z))^* = h(1/z^*)$ and so, for $z = e^{i\sigma}$, $h(z)$ is a real-valued function. Then, from (A.11), we find that $v(e^{i\sigma})$ is real-valued. Consequently, from (A.3), we find that $|f(z)| = 1$ for $|z| = 1$.

For $z = e^{i\sigma}$, we write the phase of $f(z)$ as $\phi(\sigma)$:

$$\phi(\sigma) = \frac{1}{i} \ln f(e^{i\sigma}). \quad (\text{A}\cdot 12)$$

Differentiating the phase with respect to σ , we have

$$\frac{d\phi(\sigma)}{d\sigma} = \frac{2e^{v(e^{i\sigma})}}{\{1 + e^{2v(e^{i\sigma})}\} e^{h(e^{i\sigma})} \cos \sigma}. \quad (\text{A}\cdot 13)$$

Since $v(e^{i\sigma})$ and $h(e^{i\sigma})$ are real, the derivative is positive for $|\sigma| \leq \pi/2$ and so $\phi(\sigma)$ is a monotonically increasing function from $-\pi/2$ to $\pi/2$. Hence, we have proved the properties (2.19) and (2.20).

Finally, we consider the inversion formula (2.21). We note that the differential equation (2.18) has symmetries under the following transformations:

$$z \rightarrow -\frac{1}{z}, \quad (\text{A}\cdot 14)$$

$$f \rightarrow \frac{af + b}{-bf + a}, \quad (a^2 + b^2 = 1, \quad a, b \in \mathbb{C}). \quad (\text{A}\cdot 15)$$

The first is a \mathbb{Z}_2 transformation derived from $h(-1/z) = h(z)$, which is needed for the identity-based solution as mentioned in the introduction. The second transformation forms the group $SO(2, \mathbb{C})$ in which $f = \pm i$ are fixed points. Therefore, if a special solution $f(z)$ is known, a general solution is given by the above $SO(2, \mathbb{C})$ transformation of $f(z)$. In

fact, $SO(2, \mathbb{C})$ has two real parameters and these correspond to integration constants for the complex first-order differential equation (2.18). Then, since $f(-1/z)$ is also a solution due to the first symmetry, we find that the relation

$$f\left(-\frac{1}{z}\right) = \frac{af(z) + b}{-bf(z) + a} \quad (\text{A.16})$$

has to hold for some $SO(2, \mathbb{C})$ parameters a, b . By performing this transformation twice, we can determine the parameters as $(a, b) = (1, 0)$ or $(0, 1)$. Consequently, since $f(z)$ is holomorphic at $z = i$ and $f'(i) \neq 0$, $f(z)$ must satisfy the inversion formula (2.21).

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